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Conservation of Mass & Angular Momentum Leading to the Equation for the Mass Surface Density Evolution of a Thin Keplerian Accretion Disk

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ABSTRACT

A pedagogical discussion and derivation of the time-dependent surface density $\Sigma(R, t)$ of a geometrically thin, optically thick accretion disk under the assumption of Keplerian circular motion is presented.

Subject headings: SDSU Astr 640; accretion disk theory

1. Mass and Angular Momentum in a Thin Accretion Disk[1]

1.1. Setting the Stage: Defining the Mass and Angular Momentum in an Annulus

Consider an axi-symmetric geometrically-thin accretion disk whose mass surface density is $\Sigma(R, t)$. Gas in the disk moves in circular orbits with the local Keplerian velocity $v_\phi(R)$. This azimuthal velocity can be re-written as $v_\phi = R\Omega(R)$ where $\Omega(R)$ is the angular velocity at radius R. Circular orbits are considered because a circular orbit has the minimum energy for a given angular momentum, and because the timescale for energy loss (via viscous "friction") is very much faster than the timescale for angular momentum re-distribution. To clarify, "energy loss" means the conversion of bulk kinetic energy into random thermal energy (i.e. heat), which then cools via radiative emission (i.e. photons carry energy away, which we detect as the luminosity of the disk). The bulk kinetic energy of course comes from the gravitational potential energy of the gas.

In addition to its orbital motion, mass in the disk also slowly flows in the radial direction with velocity $v_r(R)$. A negative value of v_r implies inflow toward the central mass. Do not confuse this slow inward radial velocity drift, $v_r(R)$, with the highly supersonic azimuthal velocity $v_\phi(R)$.

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[1] Throughout this document the use of parentheses indicates a functional dependence, as in $f(x)$, while brackets will be used to indicate grouped quantities, as in $[G(f) * H(f)]$. Also, for simplicity, vector notation has been dropped, e.g., $\vec{J} = \vec{R} \times m\vec{v}$ is simply $J = Rmv$, since for all cases considered here $\vec{J}$ is perpendicular to the radial coordinate $\vec{R}$ and $\vec{v}_r$ is (anti)parallel to R. Note that the symbol "×" is used for clarity in the derivations and denotes a simple multiplication, not a vector cross product.

An annulus in the disk with radius R and radial width dR will have a mass given by:

$$M = area \times surface\ density = 2\pi R dR \times \Sigma(R,t)$$

The angular momentum J of the annulus is:

$$\begin{aligned} J &= mass \times angular\ velocity = M \times R^2\Omega(R) \\ &= [2\pi R dR \Sigma(R,t)] \times R^2\Omega(R). \end{aligned}$$

1.2. Conservation of Mass

Mass will flow radially into and out of this annulus, so the mass in the annulus will not necessarily be constant. But conservation of mass certainly applies so we can write the following:

$$\frac{\partial M}{\partial t} = Mass_{\text{ into annulus}} - Mass_{\text{ out of annulus}}$$

The left-hand side, $\frac{\partial M}{\partial t}$, is simply $2\pi R dR \frac{\partial}{\partial t}\Sigma(R,t)$ because R and $\Omega(R)$ are *not* functions of time and are therefore are constants in the derivative. Remember, R is the radial coordinate and Ω is the orbital angular frequency — both are independent of time (e.g. at a specific radius in the disk, the orbital velocity is not speeding up or slowing down). The radial velocity v_r is a function of time, as well as radius. Indeed, one of the goals of this discussion is to derive the equation that gives the time evolution of v_r.

The right-hand side is a little more complicated. First, realize that the mass drifting into the annulus is coming from a radius $R + dR$, while the mass leaving the annulus is exiting at radius R. Also, the direction of the mass flow, towards the central mass, is opposite to the sense in which the radial distance is measured. In other words, the direction of $\vec{v}_r(R)$ is opposite to $\vec{R}$, and so if we replace $\vec{v}_r(R)$ with simply $v_r(R)$, we have to include a negative sign.

Fig. 1 is a cartoon illustration of the mass flow into the annulus. The top panel shows the situation in 3-D, which for many of us is easier to visualize. The amount of mass crossing into the annulus depends on the density ρ × the volume V that can cross the boundary in a time Δt. In the upper panel of Fig. 1 a small, local part of the disk is shown where a rectangular volume is drifting in the radial direction toward the central mass far off on the right. The volume depends on the width (x) × length (y) × height (z), where the width is defined to be along the radial direction. The width is just the velocity of the radial flow $v_r(R)$ × the time interval Δt. In the lower panel the 2-D case is illustrated, and the situation is exactly analogous: the mass transported across the boundary is the surface density Σ × the area that crosses the boundary into the annulus. The size of the area is the width × the length where, as before, the width is $x = v_r(R)\Delta t$. If we consider the mass flowing into the annulus per unit time, we must divide by Δt.

The above discussion considered only a local part of the disk. To consider the entire disk, we have to extend the length, y, around the circumference of the disk. Hence $y = 2\pi R$. See Fig. 2 for the corresponding illustration. Thus the mass flowing into the annulus is $Mass_{in} = v_r \Delta t \times 2\pi R \times \Sigma$. But remember that the inflowing mass is coming from radius $R + dR$, not just R. Treating the outgoing mass in exactly the same fashion, we have per unit time:

$$\begin{aligned}
&= \text{Mass}_{\text{ in}} - \text{Mass}_{\text{ out}} \\
&= (-)\frac{1}{\Delta t}\left\{\left[v_r(R+dR)\Delta t \times 2\pi\left[R+dr\right]\Sigma(R+dr)\right] - \left[v_r(R)\Delta t \times 2\pi R\Sigma(R)\right]\right\} \\
&= -2\pi\left\{v_r(R+dR)\left[R+dr\right]\Sigma(R+dr) - v_r(R)R\Sigma(R)\right\}
\end{aligned}$$

The negative sign (–) in the second line is required because the mass flowing inwards has the opposite direction of the radial coordinate R.

Now notice that the expression in braces { } is almost identical in structure to the definition of the derivative: $\frac{\partial f}{\partial x} = \frac{f(x+\Delta x)-f(x)}{\Delta x}$ in the limit that $\Delta x \to 0$. So by multiplying the term in braces by dR, the right-hand side becomes: $-2\pi dR\frac{\partial}{\partial R}(v_r R\Sigma)$.

So then going back to get the left-hand side of the equation $\frac{\partial}{\partial t}M$, we have:

$$\left[\frac{\partial}{\partial t}M = 2\pi R dR\frac{\partial}{\partial t}\Sigma(R,t)\right] = -2\pi dR\frac{\partial}{\partial R}(v_r R\Sigma)$$

canceling some terms leaves:

$$\frac{\partial}{\partial t}\Sigma(R,t) = \frac{-1}{R}\frac{\partial}{\partial R}(v_r R\Sigma).$$

This is the conservation of mass equation. This is how the mass surface density as a function of radius evolves with time.

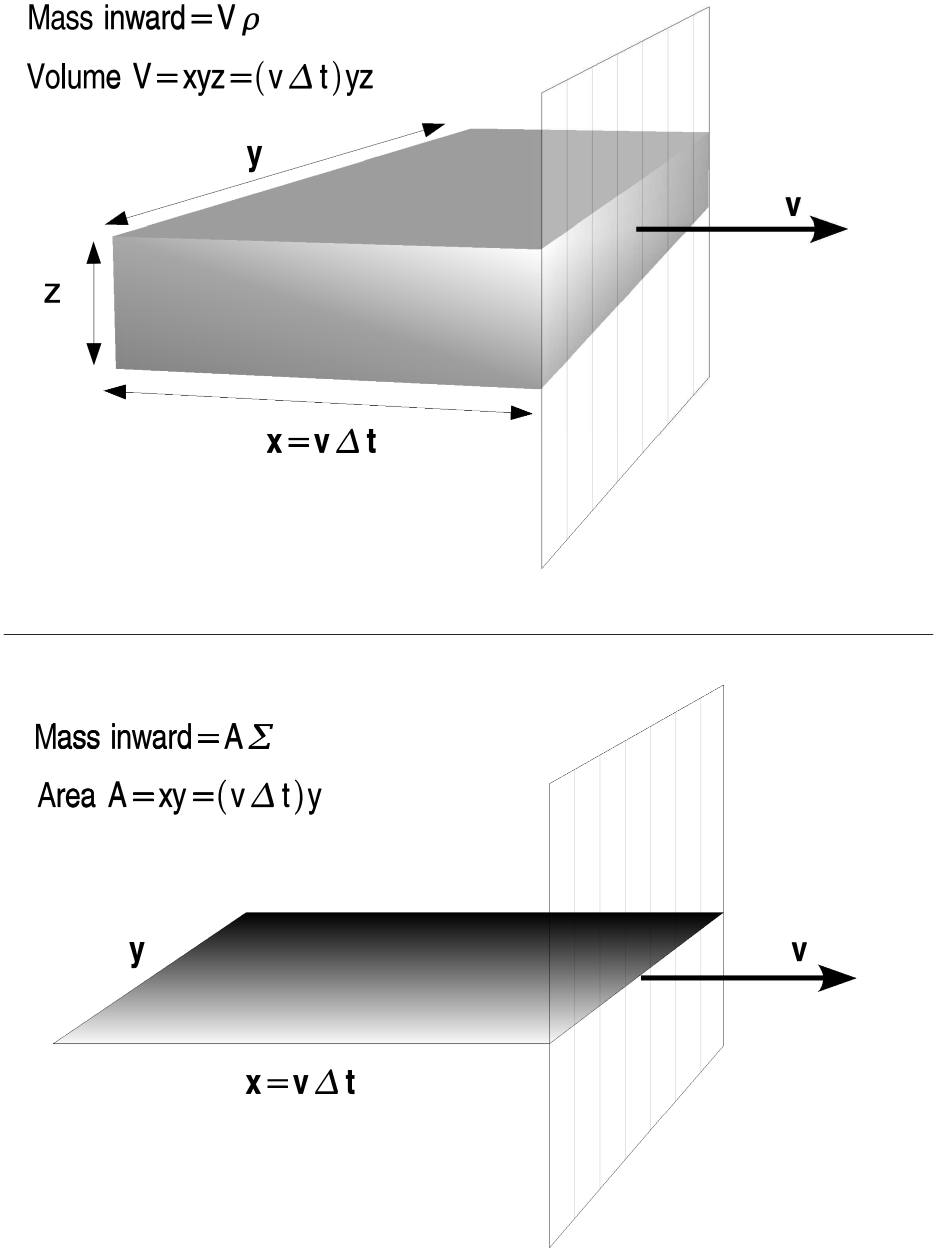
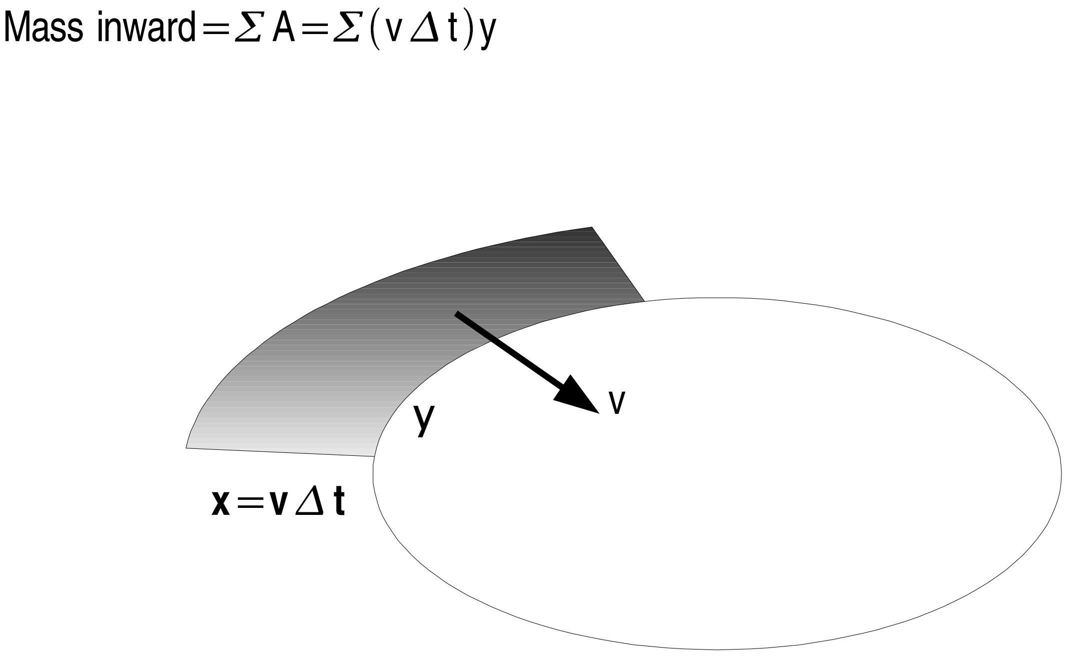


Fig. 1.— Schematic diagram of a small, local region of an accretion disk where mass is drifting inward from left to right. The mass contained in a volume V crosses the outer boundary into the annulus as discussed in the text. The upper panel shows the 3-D case and the lower panel shows the 2-D case, appropriate for a thin accretion disk. Note that $x \neq dR$; dR is the width of the annulus and is not shown in the figure.



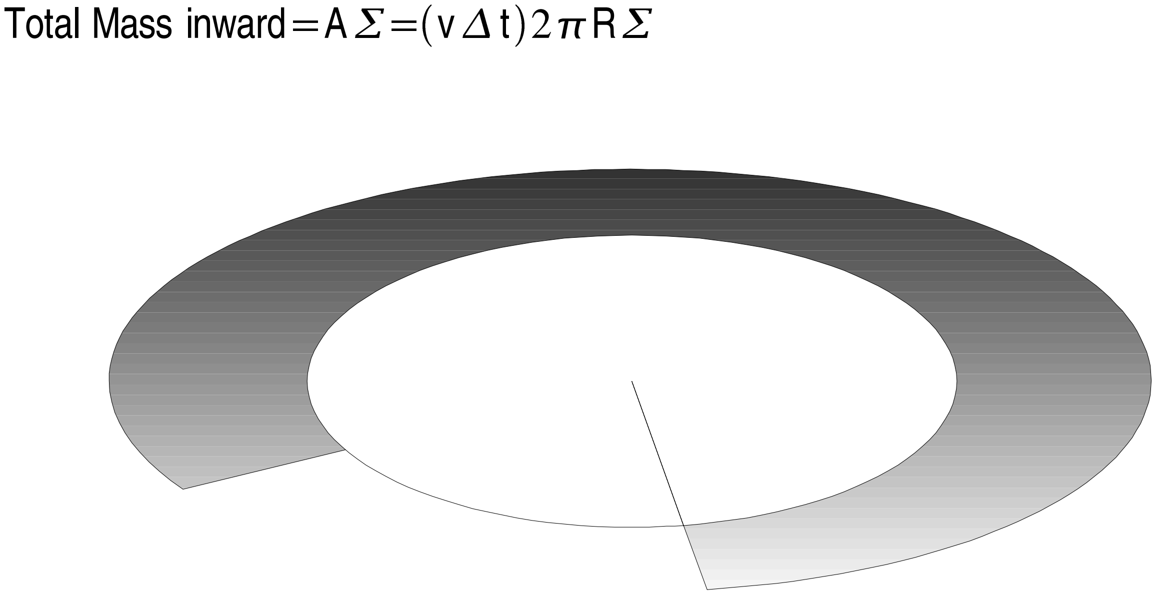


Fig. 2.— Detail of radial mass flow in a thin disk, were the local picture is replaced by the global view of the disk. (For consistency of notation, the parentheses in the figures should really be brackets.)

1.3. Conservation of Angular Momentum

For the angular momentum, the same derivation mechanics hold as for the conservation of mass, except that we must include $G_t(R)$, the viscous torque's effect on J. The subscript t on G_t stands for "torque" and is there to help prevent confusion with the gravitational constant.

So we have:

$$\begin{aligned}
\frac{\partial}{\partial t}J &= [J_{in}] - [J_{out}] + \textit{net torque on ring} \\
\frac{\partial}{\partial t}\left\{2\pi R dR \Sigma(R,t) R^2 \Omega(R)\right\} &= [J_{in}] - [J_{out}] + dR\frac{\partial}{\partial R}G_t \\
2\pi R dR R^2 \Omega(R) \frac{\partial}{\partial t}\Sigma(R,t) &= (-)\left\{\left[2\pi[R+dr]v_r(R+dr)\Sigma(R+dR)[R+dR]^2\Omega(R+dR)\right]\right. \\
&\quad \left. - \left[2\pi R v_r(R)\Sigma(R)R^2\Omega(R)\right]\right\} \\
&\quad + dR\frac{\partial}{\partial R}G_t
\end{aligned}$$

As with the conservation of mass, the (–) is required. Notice that the first two terms on the right hand side are akin to the derivative with respect to R. So replacing those terms with $dR\frac{\partial}{\partial R}()$ and then dividing both sides by $2\pi dR$ we have:

$$R^3\Omega(R)\frac{\partial}{\partial t}\Sigma(R,t) = -\frac{\partial}{\partial R}(R v_r \Sigma R^2 \Omega) + \frac{1}{2\pi}\frac{\partial}{\partial R}G_t$$

This is the expression for the conservation of angular momentum.

1.4. The Radial Drift Velocity

Breaking the first term on the right-hand side up into $\frac{\partial}{\partial R}([R v_r \Sigma]\left[R^2\Omega\right])$ then applying the product rule we get $\left[R^2\Omega\right]\frac{\partial}{\partial R}(R v_r\Sigma) + [R v_r \Sigma]\frac{\partial}{\partial R}(R^2\Omega)$. Substituting this into the conservation of angular momentum equations gives:

$$R^3\Omega\frac{\partial}{\partial t}\Sigma(R,t) = -\left[R^2\Omega\right]\frac{\partial}{\partial R}(R v_r \Sigma) - [R v_r \Sigma]\frac{\partial}{\partial R}(R^2\Omega) + \frac{1}{2\pi}\frac{\partial}{\partial R}G_t$$

Dividing both sides by R and moving the first term on the right-hand side to the left and then factoring out $R^2\Omega$ yields:

$$R^2\Omega\left[\frac{\partial}{\partial t}\Sigma(R,t) + \frac{1}{R}\frac{\partial}{\partial R}(R v_r \Sigma)\right] = -[v_r\Sigma]\frac{\partial}{\partial R}(R^2\Omega) + \frac{1}{2\pi R}\frac{\partial}{\partial R}G_t$$

But notice that the expression in [brackets] on the left-hand side is really just the conservation of mass equation, so it must $\equiv 0$. Hence we have

$$
\begin{aligned}
0 &= -\left[v_r\Sigma\right]\frac{\partial}{\partial R}(R^2\Omega) + \frac{1}{2\pi R}\frac{\partial}{\partial R}G_t \\
0 &= -\left[v_r\Sigma\right]\left[2R\Omega + R^2\frac{\partial}{\partial R}\Omega\right] + \frac{1}{2\pi R}\frac{\partial}{\partial R}G_t \\
v_r\Sigma\left[2R\Omega + R^2\frac{\partial}{\partial R}\Omega\right] &= \frac{1}{2\pi R}\frac{\partial}{\partial R}G_t \\
v_r(R) &= \left\{\Sigma\left[2R\Omega + R^2\frac{\partial}{\partial R}\Omega\right]\right\}^{-1}\frac{1}{2\pi R}\frac{\partial}{\partial R}G_t
\end{aligned}
$$

This is the equation for the radial drift velocity of the accretion flow in the disk.

1.5. Applying the Keplerian Condition

Now let's apply the Keplerian motion assumption into the above expression for v_r. Recall that $v_\phi = \sqrt{GM_1/R}$, where M_1 is the mass of the accreting object and G is the gravitational constant. Since $\Omega(R) \equiv v_\phi(R)/R$, we have $\Omega(R) = \sqrt{GM_1}\ R^{-3/2}$; and so

$$\frac{\partial}{\partial R}\Omega(R) = \frac{-3}{2}\sqrt{GM_1}\ R^{-5/2}.$$

Inserting this into the above expression for $v_r(R)$ gives:

$$
\begin{aligned}
v_r(R) &= \left\{\Sigma\left[2R\sqrt{GM_1}\ R^{-3/2} + R^2\frac{-3}{2}\sqrt{GM_1}\ R^{-5/2}\right]\right\}^{-1}\frac{1}{2\pi R}\frac{\partial}{\partial R}G_t \\
v_r(R) &= \left\{\Sigma\sqrt{GM_1}\left[2R^{-1/2} - \frac{3}{2}R^{-1/2}\right]\right\}^{-1}\frac{1}{2\pi R}\frac{\partial}{\partial R}G_t \\
v_r(R) &= \left\{\Sigma\sqrt{GM_1}\left[\frac{1}{2}R^{-1/2}\right]\right\}^{-1}\frac{1}{2\pi R}\frac{\partial}{\partial R}G_t \\
v_r(R) &= \frac{1}{\sqrt{GM_1}\ \Sigma R^{1/2}}\frac{1}{\pi}\frac{\partial}{\partial R}G_t
\end{aligned}
$$

Now let's look at the net viscous torque $G_t(R)$ term. From an earlier derivation, $G_t(R) = 2\pi R\nu\Sigma R^2\Omega'$. Recall that the prime notation ($'$) denotes the derivative with respect to R. Using the Keplerian conditions, we then have

$$
\begin{aligned}
G_t(R) &= 2\pi R\nu\Sigma R^2\left[\frac{-3}{2}\sqrt{GM_1}\ R^{-5/2}\right] \\
G_t(R) &= -3\pi\nu\sqrt{GM_1}\Sigma R^{1/2}.
\end{aligned}
$$

Now plugging this expression for $G_t(R)$ into the expression for $v_t(R)$, we have at last:

$$\begin{aligned} v_r(R) &= \frac{1}{\sqrt{GM_1}\ \Sigma R^{1/2}} \frac{1}{\pi} \left\{ \frac{\partial}{\partial R} \left(G_t(R) \right) \right\} \\ v_r(R) &= \frac{1}{\sqrt{GM_1}\ \Sigma R^{1/2}} \frac{1}{\pi} \left\{ \frac{\partial}{\partial R} \left(-3\pi\nu\sqrt{GM_1}\Sigma R^{1/2} \right) \right\} \\ v_r(R) &= \frac{-3}{\Sigma R^{1/2}} \frac{\partial}{\partial R} \left(\nu \Sigma R^{1/2} \right) \end{aligned}$$

This is the radial inflow velocity of mass in a thin Keplerian accretion disk.

1.6. The Punch Line: Time–Dependent Mass Flow

Armed with the expression for the radial inflow velocity, we now go back to the conservation of mass equation and substitute in for v_r:

$$\begin{aligned} \frac{\partial}{\partial t}\Sigma(R,t) &= \frac{-1}{R}\frac{\partial}{\partial R}(v_r R \Sigma) \\ \frac{\partial}{\partial t}\Sigma(R,t) &= \frac{-1}{R}\frac{\partial}{\partial R}\left(\left[\frac{-3}{\Sigma R^{1/2}} \frac{\partial}{\partial R} \left(\nu \Sigma R^{1/2} \right) \right] R\Sigma \right) \end{aligned}$$

which upon simplification becomes:

$$\frac{\partial}{\partial t}\Sigma(R,t) = \frac{3}{R}\frac{\partial}{\partial R}\left\{ R^{1/2} \frac{\partial}{\partial R} \left(\nu \Sigma R^{1/2} \right) \right\}.$$

This is the fundamental differential equation governing time–dependent flow of mass in a Keplerian accretion disk. This non-linear diffusion equation describes the temporal surface density evolution.

Determining the solution to this equation is non-trivial even in the most simple case, and we won't attempt it here. Indeed, you may notice that the right-hand side contains the expression for the viscosity ν, which is unknown and in general could be a function of radius. So we cannot proceed without some assumption about the viscosity.

Nevertheless, once a functional form of $\nu(R)$ is specified, the above equation can be solved and the resulting behavior of Σ is simple to understand. Being a diffusion equation, we can intuitively understand that the mass in an annulus in the disk will spread out in time. The interesting and important thing is that the mass flows both inward and outward but not in equal amounts: the vast bulk of the mass flows radially inward while a small amount of mass flows outward. The outward radial flow, even though of low mass, carries nearly all the angular momentum with it. This allows the inward flow to occur — otherwise conservation of angular momentum would prohibit an inward (or outward) net radial flow. Note that angular momentum remains conserved — it is just re-distributed. The transportation of angular momentum outward allows the bulk of the mass to drift inward, dropping lower in the gravitational potential well. And thus the accretion disk allows gravitational potential energy to be transformed into bulk orbital kinetic energy and, because of the kinematic viscosity and the shearing flow, some of the kinetic energy becomes thermal energy. ("Some" is approximately 50%, á la the virial theorem.) The gas in the disk is heated, and because the gas density is high (very optically thick), the disk radiates efficiently producing a large luminosity. The luminosity of each annulus is proportional to the local temperature $L(R) \propto \sigma T^4(R)$. Thus each annulus will radiate as a blackbody whose temperature is a function of radius. But what is that function $T(R)$? It is not hard to derive from what we have already done, keeping in mind that there must be a boundary layer where gas makes the transition from rapid orbital motion in the disk to becoming part of the more slowly rotating star. But let's jump straight to the result:

$$T(R) = \left\{ \frac{3GM\dot{M}}{8\pi\sigma R^3} \left[1 - \left(\frac{R_*}{R} \right)^{1/2} \right] \right\}^{1/4}$$

The key point is that for $R \gg R_*$, the temperature drops off as $T(R) \propto R^{-3/4}$. This is a fundamental prediction of this accretion disk theory.

Testing this prediction via observations is difficult, but it has been done using the *eclipse mapping method* applied to cataclysmic variable (CV) star disks. Eclipse mapping was invented by Keith Horne, and some of the very best work was done by Janet Wood. Eclipse mapping uses the maximum entropy method to turn the light curve into a map of the brightness temperature of the accretion disk. The observations indicate that for CV disks in their high-viscosity, high-$\dot{M}$ state (like the novalike systems), the disk temperature profile does appear to have a $R^{-3/4}$ radial dependence. While the theory is far from complete, this observational result lends confidence that our theory and the assumptions that went into it (e.g. the assumption that the gas radiates like a blackbody) are reasonably valid. It should be noted that systems that are not in the high-$\dot{M}$ state (e.g. dwarf novae in quiescence) do not show a $R^{-3/4}$ radial dependence; they tend to have a flatter temperature distribution. A weaker prediction is that the SED should have a $f_\nu \propto \nu^{1/3}$ component at intermediate frequencies if the disk is very large ($R_{out} \gg R_{in}$). However, the spectrum from a disk system contains many other components (e.g. light from the stars in a binary system, boundary layer, disk chromosphere, disk wind, disk corona, jets, irradiation effects, and the BLR in AGN), so the $\nu^{1/3}$ signature is often not apparent.

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See also: Kolb, Ulrich 2010 *Extreme Environment Astrophysics* Camridge University Press

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